Hilbert Space Quantum Mechanics Is Noncontextual

Robert B. Griffiths *
Department of Physics, Carnegie-Mellon University,
Pittsburgh, PA 15213, USA

Version of 6 January 2012

Abstract

It is shown that quantum mechanics is noncontextual if quantum properties are represented by subspaces of the quantum Hilbert space (as proposed by von Neumann) rather than by hidden variables. In particular, a measurement using an appropriately constructed apparatus can be shown to reveal the value of an observable A possessed by the measured system before the measurement took place, whatever other compatible ($[B, A] = 0$) observable B may be measured at the same time.

Contents

1 Introduction

The question of whether quantum mechanics is in some sense "contextual" seems to have first been raised by Bell, Sec. 5 of [1], in the course of a discussion of the possible form which might be taken by hidden variables in quantum theory, i.e., variables which are in addition to the Hilbert space customarily used to discussing quantum mechanics. The basic idea, not altogether easy to extract from Bell's paper, is the following (Ch. 7 of [2] or Sec. VII of [3]): Let A, B, C be three observables, represented by Hermitian operators on a finite-dimensional Hilbert space, for some system which is to be measured, and suppose that A commutes with B and with C, but B and C do not commute with each other. Then according to standard quantum mechanics it is possible in principle to measure A and B together in a single experiment, and it is possible to measure A and C together, though all three cannot be measured simultaneously. Would a measurement of A when carried out along with a measurement of B yield the same result (for A) as a measurement of A carried out along with a measurement of C?

The answer would seem to be an obvious "Yes" if the measurement, however it is carried out, reveals a value of A that was there before the measurement took place. We shall argue that this is, indeed, correct if the apparatus has been designed and operated by a competent experimentalist. However, a substantial literature has accumulated which would throw doubt on this or suggest the opposite on the basis of various arguments

*Electronic mail: rgrif@cmu.edu

related to the Bell-Kochen-Specker theorem [1,4]. A recent discussion by Hermens [5] provides an extensive list of references to work published since 2000, to which could be added much additional material that has appeared in the last two years. How have so many come to this conclusion? By adopting, we shall argue, the view that the real world is *classical*, contrary to all we have learned from the development of quantum mechanics in the twentieth century. In particular, discussions couched in terms of hidden variables typically assume that they are classical rather than the sort of thing one might expect in a quantum mechanical world.

Such confusion is not unrelated to the failure of introductory quantum mechanics textbooks and courses to provide a proper discussion of what actually happens during a quantum measuring process. Rather than explaining how a piece of apparatus interacts with and thereby extracts and amplifies information present in the measured system, students are given calculational tools, such as wave function collapse, whose connection with the quantities supposedly measured is at best vague and sometimes quite misleading. When Bell introduced the idea of contextuality in 1966 there was no satisfactory theory of quantum measurement processes, and so it was reasonable to ask whether measurement outcomes revealed values of some sort of hidden variable(s). But developments since then, beginning in the 1980s [6–8], allow measurements and other quantum processes to be described in a fully consistent way using only the quantum Hilbert space. A very detailed presentation of this (consistent or decoherent) histories approach will be found in [9], and shorter treatments in [10–12], while various conceptual issues and criticisms are discussed in [13].

The remainder of this paper is organized as follows. Section 2 is a review of quantum properties and variables as represented in a Hilbert space. Measurements are the subject of Sec. 3. They are discussed using a quite general and fully quantum mechanical measurement model involving no hidden variables, with the conclusion that an apparatus properly designed to reveal the value that the observable A had just before the measurement took place will do just that, and this remains true it the apparatus also measures a compatible (i.e., $[B, A] = 0$) observable B along with A. This settles the contextuality question for Hilbert space quantum mechanics. The final Sec. 4 summarizes the result and considers its relation to hidden variables approaches.

2 Quantum Properties and Variables

We begin with a brief review of how the quantum properties of a system s are represented using a Hilbert space $\mathcal{H}_s$ which (as in Bell's original discussion) can be conveniently assumed to be finite dimensional. An *observable* or *quantum variable* is represented by a Hermitian operator $A = A^\dagger$ on $\mathcal{H}_s$ with spectral representation

$$A = \sum_\alpha a_\alpha P_\alpha, \tag{1}$$

where the $\{a_\alpha\}$ are the eigenvalues of A, and the $\{P_\alpha\}$ a collection of projectors that form a *decomposition of the identity* I:

$$I = \sum_\alpha P_\alpha, \quad P_\alpha = P_\alpha^\dagger = P_\alpha^2, \quad P_\alpha P_{\alpha'} = \delta_{\alpha\alpha'} P_\alpha. \tag{2}$$

We assume that $a_\alpha \neq a_{\alpha'}$ whenever $\alpha \neq \alpha'$, so the decomposition $\{P_\alpha\}$ is uniquely determined by A.

A projector $P = P^\dagger$ projects onto a linear subspace of $\mathcal{H}$, and there is a one-to-one correspondence between projectors and subspaces, so one can speak either of a collection of subspaces or of projectors; we shall generally employ the latter. Following von Neumann, Sec. III.5 of [14], we think of a subspace, or equivalently the associated projector, as the mathematical representation of a quantum *property*, the quantum analog of a collection of points E in a classical phase space representing a classical property such as "the energy is less than 2 Joules." (The quantum analog of a single point in a classical phase space is a one-dimensional subspace (a ray) consisting of complex multiples of any of its nonzero elements.)

Thus for any observable A there is unique subspace P_α of $\mathcal{H}$ where A takes on its eigenvalues a_α; it consists of all the eigenvectors of A having this eigenvalue, together with the zero vector. In classical mechanics if a system is described by a point in the phase space inside the set E corresponding to some property we say that the system *has* or *possesses* the property E. Similarly, in quantum theory it makes sense to say that if a system is described by a nonzero $|\psi\rangle$ in the subspace P_α, which is to say

$$P_\alpha|\psi\rangle = |\psi\rangle, \tag{3}$$

then the system has the property P_α. (The reason for requiring that $|\psi\rangle$ be nonzero is that the zero ket, which lies in every subspace, is the quantum analog of the the empty set in a classical phase space: it

represents a proposition or property which is never true.) If the quantum system is described by a density operator ρ such that

$$P_\alpha \rho = \rho = \rho P_\alpha = P_\alpha \rho P_\alpha, \tag{4}$$

where any one of the equalities implies the others, it again makes sense to say that the system has the property P_α.

The decomposition $\{P_\alpha\}$ is the quantum analog of a coarse graining of the classical phase space into a collection of nonoverlapping cells, and represents a quantum *sample space* in the sense employed in probability theory: a set of mutually-exclusive properties ("events"), one and only one of which is true, i.e., actually occurs, at a particular time. Since A is, in effect, a real-valued function on this sample space it can be thought of as a quantum analog of a "random variable" in probability theory, or of a real-valued function on the classical phase space. Thus in what follows we shall sometimes refer to a Hermitian operator A as a "physical variable," in addition to using the traditional term "observable."

The quantum-classical analogy cannot be pushed too far, since quantum theory allows for noncommuting observables, whereas in classical mechanics all physical variables (functions on the phase space) commute. We shall call the observables A and B *compatible* when $[A, B] = 0$; otherwise they are (mutually) *incompatible*. Similarly, two projectors P and Q are compatible if $[P, Q] = 0$, in which case the product $PQ = QP$ is itself a projector onto a subspace which represents the conjunction P AND Q of the properties P and Q. Otherwise they are incompatible, and defining the conjunction P AND Q is problematical. In traditional quantum logic, descending from the work of Birkhoff and von Neumann [15], P AND Q is the property associated with the subspace which is the intersection of the two subspaces corresponding to P and Q. This, however, requires the use of new logical rules to avoid falling into difficulties—for a simple example see Sec. 4.6 of [9]. But the main complaint about quantum logic is that it has not resolved the major conceptual difficulties of quantum mechanics [16, 17]. In the histories approach one gets around these difficulties by insisting that P AND Q only makes sense, can only be discussed, when the projectors commute. This is an example of the *single framework rule*, a principle which can be used to resolve numerous quantum paradoxes, as discussed in Chs. 20 to 25 of [9].

Consider an observable B that is compatible with A, with a spectral representation

$$B = \sum_\beta b_\beta Q_\beta, \tag{5}$$

where again the decomposition of the identity $\{Q_\beta\}$ is chosen so that $b_\beta \neq b_{\beta'}$ whenever $\beta \neq \beta'$. The fact that $[A, B] = 0$ implies that $[P_\alpha, Q_\beta] = 0$ for every α and β, and consequently there is a third decomposition of the identity $\{R_j\}$, the common refinement of $\{P_\alpha\}$ and $\{Q_\beta\}$, consisting of all nonzero projectors of the form $P_\alpha Q_\beta$, such that

$$A = \sum_j a_j R_j, \quad B = \sum b_j R_j. \tag{6}$$

The a_j in (6) are of course eigenvalues of A, but now two or more of them may have the same value, in contrast to (1), which is the reason for using a distinct subscript; the same comment applies to the b_j. Because it consists of products of the form $P_\alpha Q_\beta$, the decomposition $\{R_j\}$ is the smallest collection of projectors such that both A and B can be written in the form (6). If $|\psi\rangle$ belongs to a subspace R_j, then since R_j is itself the product $P_\alpha Q_\beta$ for some specific choice of α and β, it follows that if the quantum system under discussion has the property R_j it also has the property P_α in the sense that (3) is satisfied, and likewise the (compatible) property Q_β. This is obvious for the analogous situation of a classical phase space, but in the quantum case it only makes sense when all the projectors that we are considering, the $\{P_\alpha\}$, $\{Q_\beta\}$, and $\{R_j\}$, commute with each other; as noted above there are logical difficulties in the noncommuting case.

It should be noted that *quantum physical variables (observables)* and *properties* as defined above above apply not only to microscopic quantities, but also to macroscopic objects, such as laboratory equipment. There is no experimental evidence for a "classical" world to which quantum principles do not apply. Quantum theory is routinely used when discussing properties of the solid materials out of which experimental apparatus is constructed, the properties of stars, etc. So far as we know at present, it applies "from the quarks to the quasars." To be sure, there are excellent reasons why engineers discuss the motion of automobiles using classical mechanics rather than quantum theory, but this is a practical matter, not one of principle. We now understand, at least in general terms, how in suitable circumstances classical physics emerges as a good approximation to a more fundamental quantum physics [18,19], and this justifies the fully

quantum mechanical description given below of a physical measurement carried out by a macroscopic piece of apparatus.

3 Measurements

3.1 Introduction

The most direct approach to determining whether quantum theory is or is not contextual is to analyze the process that goes on in a quantum measurement. A measurement, as that term is ordinarily employed in physics, means determining one or more properties possessed by a system at a time just preceding that at which the measurement was carried out. It is in this sense that experimental particle physicists can speak, as they do, of detecting neutrinos arriving from the sun, or measuring the energy of a particle shower that has come to rest in their instrument. The reason for referring to the time just *before* the measurement is carried out is that the measuring process itself may change the property in question, and sometimes, as in the case of photons, the object being measured actually disappears. There is an important difference between a measurement and a *preparation* in which the instrument and setup tells one something about the properties of the prepared system *after* the preparation takes place. Considerable confusion is created in discussions of quantum foundations, and in the minds of students studying quantum theory for the first time, by the common failure to make a clear distinction between measurement and preparation. Part of this goes back to von Neumann's treatment of measurements, Ch. VI of [14], in which he employed a very special model that in effect combines a measurement with a preparation. Whatever the utility of this model when quantum theory was first developed, greater clarity is possible nowadays if the distinction between these processes is kept in mind [13].

3.2 Measurement model

We use a model of the measuring process in which the apparatus itself is a fully quantum mechanical object subject to exactly the same physical laws as the system being measured. Let $\mathcal{H} = \mathcal{H}_s \otimes \mathcal{H}_m$ be the combined Hilbert space of the system s being measuring and the apparatus m. The latter can include the environment of the apparatus and whatever else is relevant to the description of the measuring process, enough so that we can consider $\mathcal{H}$ the Hilbert space of a closed system to which Schrödinger's equation applies. That equation gives rise to a unitary time evolution operator[1] $T(t_2, t_1)$ on $\mathcal{H}$, where t_1 is a time just before the measurement takes place and t_2 a time just after it is complete, with results indicated by the position of the traditional pointer. Let us assume that at the initial time the system has the property P_α and the measuring apparatus (plus environment, etc.) is in an initial macroscopic "ready" state corresponding to the property or projector M_0 onto some subspace (typically of extremely high dimension) of $\mathcal{H}_m$. Thus at t_1 the combined system plus apparatus has the property $P_\alpha \otimes M_0$. (The reader who prefers a density operator for the apparatus is welcome to employ that rather than M_0 in the following discussion.) Next we assume that there is a decomposition $\{\Pi_\alpha\}$ of the identity I for $\mathcal{H}$ in which the index α labels macrostates corresponding to different possible positions of the pointer indicating the measurement outcome, and in addition to the values in (1) there is another, say $\alpha = 0$, such that

$$\Pi_0 = I - \sum_{\alpha \neq 0} \Pi_\alpha. \tag{7}$$

The projector Π_0 allows for all other possibilities; e.g., the apparatus was not properly set up in the first place. Next we assume that the unitary dynamics is such that, defining

$$V_\alpha := T(t_2, t_1)\big(P_\alpha \otimes M_0\big)T(t_1, t_2), \tag{8}$$

it is the case that

$$\Pi_{\alpha'} V_\alpha = \delta_{\alpha\alpha'} V_\alpha. \tag{9}$$

In words, if the system has property P_α at time t_1, then the time-evolved property V_α is contained in the subspace onto which Π_α projects, meaning that the pointer is in the position α. The $\delta_{\alpha\alpha'}$ implies that the

[1] If the total system Hamiltonian H is independent of time, $T(t_2, t_1) = e^{-i(t_2-t_1)H/\hbar}$.

system property P_α at t_1 and the pointer at Π_α at t_2 are perfectly correlated: from the latter one can infer the former, and vice versa. Notice that the Π_α are projectors on $\mathcal{H}$, not on $\mathcal{H}_m$ alone, and are associated with pointer positions. They say nothing at all about the final state of the measured system s, which may not even exist. We are, as noted earlier, discussing a measurement, not a preparation.

The reader unfamiliar with this type of measurement model may find it helpful to consider the special case in which $M_0 = |M_0\rangle\langle M_0| = [M_0]$, where we use the abbreviation $[\psi] = |\psi\rangle\langle\psi|$ for the projector onto the ray containing $|\psi\rangle$, meaning the apparatus is in an initial pure state $|M_0\rangle$, and the system to be measured is in a pure state $|a_\alpha\rangle$ for some α, so $P_\alpha = [a_\alpha]$. Then if we define

$$|V_\alpha\rangle := T(t_2, t_1)\big(|a_\alpha\rangle \otimes |\Psi_1\rangle\big), \tag{10}$$

V_α in (8) is $[V_\alpha]$, and (9) is equivalent to

$$\Pi_{\alpha'}|V_\alpha\rangle = \delta_{\alpha\alpha'}|V_\alpha\rangle, \tag{11}$$

which is analogous to (3): when $\alpha = \alpha'$ the total system has the property that the pointer is in position α.

Obviously this is an idealized description of an actual laboratory measurement, but it is nonetheless accurate in representing the essential features of the latter, in somewhat the same way in which a diagram of a telescope showing rays entering the aperture before bouncing off mirrors and exiting through a lens represents the essential idea of how it operates. It is the task of the competent experimentalist to arrange a piece of apparatus so that the transition from t_1 to t_2 takes place in way such that (9) is satisfied. He can test whether or not it functions according to design by making repeated runs in which the apparatus m always starts off in the ready state M_0, and the system s is prepared in one of the states P_α, where α varies from run to run. If the final outcome (pointer position) always corresponds to the known initial state of the system, this indicates that (9) is satisfied.

3.3 Superposition states

What will occur if the experimenter prepares an initial state

$$|\psi\rangle = (|a_1\rangle + |a_2\rangle)/\sqrt{2}, \tag{12}$$

which is a superposition corresponding to distinct eigenvalues of a_1 and a_2 of A? The answer is that the outcome will differ from run to run. Sometimes the pointer (or its electronic counterpart in a modern laboratory) will have the property corresponding to Π_1, and sometimes that corresponding to Π_2, with roughly half the runs indicating one and half indicating the other. We agree with Appleby [20] that probabilities can only come from probabilities, so in order to obtain the probabilities of standard quantum mechanics one has to introduce a probabilistic axiom. The simplest is the Born rule, which for present purposes can be worded as follows. Suppose that at t_1 we assign a property $\hat{P}$ to s, where $\hat{P}$ projects on any subspace of $\mathcal{H}_s$, and define

$$\hat{V} := T(t_2, t_1)\big(\hat{P} \otimes M_0\big)T(t_1, t_2). \tag{13}$$

At t_2 we are interested in a collection of properties of the apparatus represented by the Π_α, which form a sample space of mutually-exclusive properties, to which Born's rule assigns conditional probabilities

$$\Pr(\Pi_\alpha \,|\, \hat{P}) = \mathrm{Tr}(\Pi_\alpha \hat{V})/\mathrm{Tr}(\hat{V}), \tag{14}$$

where the denominator on the right side normalizes the probability. If $\hat{P} = P_{\alpha'}$, then $\Pr(\Pi_\alpha \,|\, \hat{P}) = \delta_{\alpha\alpha'}$, consistent with the result in Sec. 3.2, but now expressed in the form of a probability.

If, however, $\hat{P} = [\psi]$ for $|\psi\rangle$ in (12), (14) will assign probabilities of 1/2 to each of the outcomes Π_1 and Π_2. What happened to Schrödinger's cat, the weird superposition of macroscopically distinct pointer positions onto which $\hat{V}$ projects? As long as we are only concerned with calculating the probabilities of the Π_α, $\hat{V}$ in (14) serves simply as a calculational tool, a pre-probability in the terminology of Sec. 9.4 of [9], and does not count as a physical property. If, on the other hand, one were to consider the decomposition of the identity formed by the projectors $\hat{V}$ and $I - \hat{V}$, the Born rule would assign a probability $\Pr(\hat{V} \,|\, \hat{P}) = 1$, the property $\hat{V}$ is certain to occur at t_2. How is this to be reconciled with the previous assignment of probabilities to Π_1 and Π_2? Observe that just as $\hat{P}$ does not commute with either P_1 or P_2, $\hat{V}$ does not commute with Π_1 and Π_2, so it makes no sense—it violates the single framework rule—to put all three together into a single discussion. (For further remarks on the single framework rule see the discussion in Sec. 2.3 of [13].)

But in addition to getting rid of the ghost of Schrödinger's cat, we still need to show that the measurement apparatus actually carries out a *measurement*; i.e., the outcome pointer position is properly correlated with a previous property of the measured system. For this purpose we need an extension of Born's rule that allows probabilities to be assigned to a closed quantum system at three or more times, and this in turn requires the use of consistent (or decoherent) families of histories. As this topic is discussed in considerable detail and with numerous examples in [9], with more abbreviated treatments in [10, 11], we omit details and give only the essential results as they apply in the present situation.

To this end it is convenient to introduce a time t_0 slightly earlier than t_1, but still later than the completion of the preparation procedure that results in the state $|\psi\rangle$ in (12). We assume that no significant time development goes on during the short interval from t_0 to t_1, and consider a family of histories

$$[\psi] \otimes M_0 \odot \{P_\alpha\} \odot \{\Pi_\alpha\}, \tag{15}$$

where the notation is to be interpreted as follows. The symbol $\odot$ separates situations at the different times; it is the counterpart of "$<$" in $t_0 < t_1 < t_2$. The initial state at time t_0 is a projector $[\psi] \otimes M_0$ with $[\psi] = |\psi\rangle\langle\psi|$ corresponding to the superposition $|\psi\rangle$ in (12). At t_1 we consider the collection of properties $\{P_\alpha\}$ making up the decomposition in (2). Similarly at time t_2 the properties under consideration are the pointer positions $\{\Pi_\alpha\}$. An extension of the Born rule allows the assignment of probabilities to the different histories, each history involving one of the properties in (15) at each time, provided appropriate consistency (or decoherence) conditions are satisfied. In the present case consistency is a consequence of (9), and the extended Born rule assigns zero probability to each history except for the pair

$$[\psi] \otimes M_0 \odot P_1 \odot \Pi_1, \quad [\psi] \otimes M_0 \odot P_2 \odot \Pi_2, \tag{16}$$

to each of which is assigned a probability of 1/2. As a consequence the conditional probability of the property P_1 at time t_1 given the pointer position Π_1 at t_1 is equal to 1, and vice versa; and the same for P_2 and Π_2. Of course there is nothing special about the superposition used in (12); any other would have led to a similar result, which is just the conclusion we reached earlier in Sec. 3.2: there is an exact, one-to-one correspondence between later pointer positions and earlier properties of the system to be measured.

In summary, a properly constructed quantum measurement apparatus does just what it was designed to do, measure properties of a system corresponding to a particular decomposition of the identity. It will do this for a system prepared in one of the states it was designed to measure, and for one prepared in a superposition of those states. To discuss the superposition case one needs to employ probabilities, in particular the (extended) Born rule, both in order to have a measurement pointer with a well-defined position, and to show that this position is appropriately correlated with one of the properties the apparatus was designed to measure. For details omitted from the preceding discussion the reader is referred to the extensive treatment of measurements in [9], as well as the shorter discussions in [10, 11, 13].

3.4 Measuring compatible observables

Next consider a situation in which two observables, A associated with the decomposition $\{P_\alpha\}$ and B with $\{Q_\beta\}$ commute with each other. As discussed in Sec. 2 there is a decomposition of the identity $\{R_j\}$ which allows both A and B to be written in the form (6), and hence to determine the values of both of them it suffices (and is also necessary) to carry out a measurement which determines which property R_j is possessed by the system. To do such a measurement we follow the same strategy as in Sec. 3.2, but now with a decomposition $\{\Xi_j\}$ corresponding to pointer positions replacing the $\{\Pi_\alpha\}$, operators

$$W_j := T(t_2, t_1)\,(R_j \otimes M_0)\,T(t_1, t_2) \tag{17}$$

in place of the V_α, and (9) replaced with

$$\Xi_j W_{j'} = \delta_{jj'} W_j. \tag{18}$$

The argument that each outcome Ξ_j reflects a prior property R_j is then the same as that used earlier.

A particular P_α may be equal to one of the R_j, or it may be the sum of two or more. Suppose, for example, that $P_1 = R_1 + R_2$. If the experimenter prepares the system with a property P_1 then it may have neither of the properties R_1 or R_2. But it will still be the case that as a consequence of the linearity of (18),

$$(\Xi_1 + \Xi_2)V_1 = V_1, \tag{19}$$

where V_1 as defined in (8). Hence, again invoking the Born rule, the measurement outcome will be one of the pointer positions corresponding to Ξ_1 or Ξ_2, varying randomly from run to run. These two outcomes are uniquely associated with the property P_1 in that they cannot occur if the initial property is P_α with $\alpha \neq 1$. Thus either outcome implies that the system earlier possessed the property P_1. Therefore this new procedure remains valid for measuring A, while at the same time, by the same sort of argument, it provides a measurement of B.

4 Discussion

The above argument establishes the noncontextuality of quantum mechanics when the measured quantities are quantum properties associated with subspaces of the Hilbert space. The reason is that for a competently constructed piece of apparatus, the measurement outcome (pointer position) of a measurement of A implies that the measured system possessed the corresponding property P_α just before the measurement took place. If the same apparatus can in addition measure a compatible observable B this does not alter the conclusion, which in no way depends on nature of the compatible variable B. In particular, replacing B with some other C for which $[C, A] = 0$ in no way affects the conclusion stated above about the measurement of A, and the possibility that one can choose variables B and C which both commute with A but do not commute with each other is completely irrelevant.

Given this fairly straightforward way of demonstrating that quantum mechanics is noncontextual, it is natural to ask why the topic of contextuality still seems to attract considerable attention, even leading to proposals of experimental tests. In this connection it is helpful to examine the recent paper by Hermens [5] already referred to in Sec. 1. He gives a list of four statements which because of the Kochen-Specker theorem cannot all be true:

> *QM* (Quantum mechanics): Every observable $\mathcal{A}$ can be associated with a self-adjoint operator A on some Hilbert space $\mathcal{H}$. The result of a measurement of $\mathcal{A}$ is an element of the spectrum $\sigma(A)$ of A. Observables whose corresponding operators commute can be measured simultaneously. Moreover, if there is a functional relationship between the associated operators, this relation is preserved in the measurement results.
>
> *Re* (Realism): Every observable $\mathcal{A}$ possesses a certain (real) value $\lambda(A)$ at all times.
>
> *FM* (Faithful measurement): A measurement of an observable $\mathcal{A}$ at a certain time reveals the value $\lambda(\mathcal{A})$ possessed by that observable at that time.
>
> *CP* (Correspondence principle): There is a bijective correspondence between observables and self-adjoint operators.

To connect these statements with our previous discussion, note that we have used the same symbol A for a physical variable or observable, and the corresponding quantum operator, which is to say we assumed the validity of Hermens' *CP* from the very outset: "observable" is just the name associated with self-adjoint or Hermitian (equivalent for finite-dimensional $\mathcal{H}$) operators in the usual textbook approach to quantum theory. We also agree with *QM*.[2] Furthermore, our measurement model, with $\lambda(\mathcal{A})$ an eigenvalue of A, satisfies *FM*, though with the qualification that the measured property was possessed by the system at a time just before the measurement took place. Thus the way our approach avoids any conflict with the Kochen-Specker theorem is by denying *Re*. The claim that *every* observable possesses a value at every time is, indeed, inconsistent with a representation of quantum properties by subspaces of a Hilbert space.

Consider, for example, a spin-half particle. There are distinct rays in the two-dimensional Hilbert space corresponding to $S_x = +1/2$ (in units of $\hbar$) and to $S_x = -1/2$; also to $S_z = +1/2$ and $S_z = -1/2$, but there is no ray that can represent a simultaneous value of S_x *and* of S_z. Students are told, correctly, that S_x and S_z cannot be measured simultaneously, and they ought to be told that the reason for this is that there is nothing there to be measured. The projectors corresponding to S_z do not commute with the projectors corresponding to S_x, and once one has accepted the connection between quantum properties and Hilbert

[2] A functional relationship between operators implies a functional relationship between the sets of eigenvalues when both are expressed using a common decomposition of the identity. E.g., one could imagine that there is a (real-valued) function $f(x)$ such that for every j in (6) $b_j = f(a_j)$. What the measurement actually reveals is the prior property R_j, but if one uses it to infer the corresponding eigenvalues the functional relationship between them will, of course, be satisfied.

subspaces proposed by von Neumann it makes no sense to speak of a spin half system in which, for example, $S_x = +1/2$ at the same time that $S_z = +1/2$. For further discussion of these matters see Sec. 4.6 of [9].

Consequently, three of Hermens' four statements (perhaps slightly modified) are in agreement with Hilbert space quantum mechanics, but *Re* is not. It is somewhat odd that this particular principle should be identified with *realism*, since at the present time all available experimental evidence is in accord with Hilbert space quantum mechanics, and not with classical physics when the two disagree. If a Hilbert space provides the appropriate mathematics to describe everything from the quarks to the quasars, where in the real world, the one we live in, is there any part that satisfies the condition of "realism" given by *Re*? It would be much less confusing if whenever "realism" were used in this way the adjective "classical" were prepended. The hidden variables of typical hidden variable theories are *classical* hidden variables, and it is for this reason that the attempt to use them for interpreting quantum theory has given rise to various conflicts with the latter.

Not only have classical hidden variables given rise to misleading notions about the contextuality of quantum measurements, they have also led to the misleading idea that quantum mechanics is nonlocal in the sense that actions at one point in spacetime have a mysterious nonlocal influence on what goes on at other, spatially separated, points in spacetime, contrary to special relativity. For a demonstration that Hilbert space quantum theory contains no such influences, see [11, 21]. It is thus surprising that classical hidden variables continue to receive so much attention in discussions of the foundations of quantum mechanics when they are or such little use. Perhaps it is time to pay serious attention to the mathematical properties of Hilbert space, and the representation of physical properties by means of its subspaces.

Acknowledgments

The research described here received support from the National Science Foundation through Grant PHY-1068331.

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